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Painlevé chains for the study of integrable higher-order differential equations

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Abstract. The concept of Painlevé chains is extended to chains of ordinary differential equations obtained by successive and simultaneous differentiation of both sides of equations of the general type $f(x, u, u_x, u_{2x}, \ldots, u_{nx}) = g(x, u, u_x, u_{2x}, \ldots, u_{mx})$ where $m \leqslant n-1$. The three Painlevé chains of my previous paper are thus generated by such successive differentiation of $u_{2x} = ku^2$ (KdV chain), $u_{2x} = ku^3$ (modified KdV chain) and $u_x = ku^2$ (Burgers chain). Hybrid Painlevé chains can be analogously obtained by successive differentiation of hybrid differential equations; such chains obtained from $u_{2x} = -uu_x + u^3$ and $u_{2x} = -3uu_x - u^3$ are described in detail. Passive differential equations in which the balancing exponent must depend upon the coefficients can also lead analogously to Painlevé chains as illustrated by those obtained from $u_{2x} = ku_x^2/u$ where $k=2$ *or* $\frac{3}{2}$. The Schwarzian derivative $(u_{3x}/u_x) - \frac{3}{2}(u_{2x}/u_x)^2$ generates a Painlevé chain in which the members have consecutive integral resonances starting with the ubiquitous -1. Chains of higher-order differential equations exhibiting some, but not all, of the features of Painlevé chains can be obtained from the second-order eigenvalue problem. The dominant truncations of most evolution equations as well as the Painlevé canonical equations including the irreducible Painlevé transcendents appear in Painlevé chains generated by the methods outlined in this paper.

1. Introduction

In recent years a certain class of non-linear higher-order partial differential equations, known as evolution equations [1-5], has become of special interest to theoretical physicists. Such equations possess a special type of elementary solution taking the form of localised disturbances which act somewhat like particles and are therefore known as solitons. These equations have applications in diverse areas of physics including fluid dynamics, ferromagnetism, quantum optics and crystal dislocations.

Solution of important evolution equations frequently involves the so-called inverse scattering transform [1-6]. In this connection the development of simple methods for identifying differential equations solvable by this approach is of interest. Thus Ablowitz *et al* [7, 8] proposed the Painlevé conjecture stating that such solvable ordinary differential equations (ODE) must have the Painlevé property, namely the location of their critical points (i.e. singularities other than poles) must be independent of the constants of integration. Subsequently Weiss *et al* [9] showed how the Painlevé property could also be defined for partial differential equations (PDE). Algorithms have been developed in order to determine whether a given ODE [8] or PDE [9] has the Painlevé property.

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In a previous paper [10] I derived three Painlevé chains which can be used to generate higher-order ordinary differential equations having necessary conditions for the Painlevé property. These Painlevé chains relate directly to the first two irreducible Painlevé transcendents [11] and the most important evolution equations including the Burgers, Korteweg-de Vries (KdV), modified KdV and Boussinesq equations. The present paper develops and extends further the concept of Painlevé chains including relationships with Schwarzian derivatives [12-15] and the second-order eigenvalue problem [3] of Zakharov and Shabat [16].

2. The Painlevé test

A solution of an ordinary differential equation may have a number of singularities which are movable or fixed depending upon whether or not their locations depend upon the constants of integration [7-9, 11]. A singularity that is not a pole (of any order) is called a critical point: such critical points may be algebraic or logarithmic branch points or essential singularities. An ordinary differential equation has the Painlevé property if it has no movable critical points. The singular point analysis for testing whether or not an ordinary differential equation of order n has the Painlevé property consists of the following three steps: (1) determination of the dominant terms of the differential equation and its balancing exponent p in a power series which characterises the behaviour of its solutions near the movable singularities; (2) solution of an indicial equation to determine the resonances $r_1, \ldots, r_n$ which indicate the terms where the integration constants can enter the above power series; and (3) determination whether the coefficients of the above power series are compatible with a pure Laurent series [17] without any logarithmic terms entering at the resonances. The Painlevé test can fail at any of these three steps as follows: (1) the balancing exponent p is not a negative integer; (2) the resonances are not integers or the indicial equation has a repeated root; or (3) the expressions for the coefficients of the power series terms at the resonances are incompatible with the identical zero values required for introduction of the integration constants. The first two steps of the singular point analysis are relatively simple since they require consideration of only the dominant terms of the ordinary differential equation, called [10] its dominant truncation. However, the third step requires the full differential equation and is thus much more tedious and complicated. The general idea behind the work discussed in this paper, as well as my previous paper [10] is to obtain the maximum information about higher-order differential equations from the first two steps of the Painlevé test, which are relatively easy. Thus the idea of Painlevé chains relates to the classification of higher-order ordinary differential equations into dominance classes having the same dominant truncations and then determining which dominance classes satisfy the first two of the three steps of the above Painlevé test.

Consider an evolution equation of the form

$$u_t + f(x, u, u_x, u_{2x}, \ldots, u_{nx}) = 0 \tag{1}$$

in which

$$u_t = \partial u/\partial t \qquad u_{jx} = (\partial/\partial x)^j u \qquad j = 0, 1, \ldots, n. \tag{2}$$

In such partial differential equations u may be regarded as an amplitude, x as a distance and t as time. Setting $u_t = 0$ leads to time-independent solutions corresponding to ordinary differential equations of the type

$$f(x, u, u_x, u_{2x}, \ldots, u_{nx}) = 0. \tag{3}$$

Let us adjust the distance scale so that $z(x) = 0$ is a critical point. The dominant behaviour of solutions of this ordinary differential equation in the neighbourhood of such a critical point can be expressed as the following series:

$$u = a_0 z^p \qquad \text{as } z \to 0. \tag{4}$$

Substitution of equation (4) into the original ordinary differential equation (equation (3)) shows that for certain values of the balancing exponent p, two or more terms may balance and the rest can be ignored as $z \to 0$. Deletion of the terms not involved in the balancing in general leads to a simpler ordinary differential equation called the dominant truncation of the original differential equation. All equations giving the same dominant truncation may be considered to form a dominance class. A self-dominant equation is one in which all of its terms are involved in the balancing and is therefore identical to its dominant truncation.

The balancing exponent p may be determined either actively or passively depending upon the differential equation in question. Active determination of p results when the exponents of the balancing terms are different expressions in p so that an equation is generated by equating the different expressions for the same exponent; solution of this equation then determines p. Passive determination of p occurs when the exponents of all of the balancing terms are the same expression in p so that they do not generate an equation to be solved for p. In these cases p must be determined from an expression arising from the coefficients of the balancing terms. In all of the differential equations discussed in the previous paper [10] the balancing exponent p is determined actively.

Now consider the dominant truncation of the ordinary differential equation in question (e.g., equation (3)) which may be represented as

$$f^*(x, u, u_x, u_{2x}, \ldots, u_{nx}) = 0. \tag{5}$$

Equation (4) may then represent the first term in a Laurent series [17] valid in a deleted neighbourhood of movable pole. In this case a solution of the original ordinary time-independent differential equation (3) is of the following type:

$$u = z^p \sum_{k=0}^{\infty} a_k z^k \qquad \text{where } z \neq 0. \tag{6}$$

The position of the singularity $z = 0$ corresponds to one of the n integration constants. If $n-1$ of the coefficients a_k are also arbitrary, the n integration constants of equation (3) are then accounted for and equation (6) represents the general solution of the time-independent equation (3) in the deleted neighbourhood of the singularity $z = 0$. The powers of z at which these arbitrary constants enter are called the resonances and will be designated as $r_1, r_2, \ldots, r_n$ so that $r_i \leqslant r_k$ and $i < k$. In a similar Laurent series expansion of the original time-independent partial differential equation (1), the coefficients a_k must be assumed to be functions of x and t rather than constants.

In order to find the resonances, the following equation for u is substituted into the dominant truncation (equation (5)):

$$u = a_0 z^p + a_r z^{p+r}. \tag{7}$$

In the usual case of equations linear in u_{nx} the coefficient a_0 is determined by equating the coefficents of the z^{p-n} terms which are the leading terms in the neighbourhood of

$z = 0$. If the balancing exponent p is determined passively, then the value of the coefficient a_0 will be arbitrary and one of the resonances will be zero. After determining a_0 then the coefficients of the next higher powers z^{p+r-n} are equated in order to determine the resonances. In this way the resulting equations for the resonances reduce to

$$Q(r) a_r z^q = 0 \qquad q \leqslant p + r - n \tag{8}$$

in which $Q(r)$ is a polynomial in r of degree n. The roots of $Q(r)$ determine the resonances since $Q(r) = 0$ corresponds to the 'indicial equation' used to solve a linear ordinary differential equation near a regular singular point [18]. One root of $Q(r)$ will always be -1 reflecting the arbitrariness of the singularity $z = 0$ corresponding to one of the n integration constants. In cases where the balancing exponent p is determined passively rather than actively, a second root of $Q(r)$ will be zero reflecting the arbitrariness of a_0 in these cases. A requirement for the Painlevé property is that all resonances $r_1, \ldots, r_n$ be distinct integers (no multiple roots). Furthermore only integers greater than -1 (i.e. zero or positive integers) indicate terms in the power series of equation (6) which can incorporate integration constants in their coefficients.

The final step of the Painlevé test consists of determining the coefficients of the power series (equation (6)) from a_0 up to the coefficient of the last resonance a_{r_n}. Because the full partial differential equation (e.g., equation (1)) must be used rather than the dominant truncation of the time-independent ordinary differential equation, this step is considerably more complicated than the first two steps and computer methods are often needed for the messy algebra [19]. The Painlevé property requires compatibility conditions to be satisfied at each of the positive integer resonances; in this case arbitrary integration constants can be introduced at each of the resonances without affecting the Laurent expansion (equation (6)). Failure to satisfy such compatibility conditions means that logarithmic terms must be introduced at the offending resonances leading to movable logarithmic branch points in violation of the Painlevé property. Since the spacing of the resonances determines the introduction of the integration constants in the power series expansion (equation (6)), the power series for solutions of differential equations having the same resonances r_i and the same balancing exponent p might be expected to exhibit some similar features. In addition, the Schwarzian Painlevé chain discussed later in this paper has a prototypical role since the resonances in members of this Painlevé chain appear consecutively from the first term on so that all of the resonances appear in the power series before any non-resonant terms. The members of the Schwarzian Painlevé chain are in the same dominance classes as higher-order differential equations based on the Schwarzian derivative used by Weiss [14] to study higher-order evolution equations including seventh-order analogues of the KdV equation.

3. The canonical Painlevé type equations

Around the turn of the century Painlevé and his coworkers examined second-order ordinary differential equations of the following form for the absence of movable critical points [11].

$$u_{2x} = F(u_x, u, x). \tag{9}$$

Assuming $F(u_x, u, x)$ to be rational in u_x and u and analytic in x, they found that equations of this type without movable critical points could be represented as one of

50 canonical types, designated [11] by the Roman numerals I to L. Among these 50 types, six could not be reduced to simpler differential equations and thus defined new transcendental functions; these irreducible second-order differential equations are the Painlevé transcendents $P_{\rm I}$, $P_{\rm II}$, $P_{\rm III}$, $P_{\rm IV}$, $P_{\rm V}$ and $P_{\rm VI}$, corresponding to the canonical forms IV, IX, XIII, XXXI, XXXIX, and L, respectively, in Ince's book [11]. The six Painlevé transcendents and some of the properties of the first four Painlevé transcendents are listed in table 1.

The dominant truncations of the 50 canonical Painlevé type equations are composed of the following four building blocks with the indicated values of the balancing exponent p:

$$u_{2x} = k_a u^2 \qquad (p = -2/1 = -2) \tag{10a}$$

$$u_{2x} = k_b u u_x \qquad (p = -1/1 = -1) \tag{10b}$$

$$u_{2x} = k_c u^3 \qquad (p = -2/2 = -1) \tag{10c}$$

$$u_{2x} = k_d u_x^2/u \qquad (p \text{ indeterminate from the exponents}). \tag{10d}$$

Since equations (10b) and (10c) have the same value of p, namely -1, hybrids [10] of the following form can be obtained from their linear combination:

$$u_{2x} = h_1 u u_x + h_2 u^3. \tag{11}$$

The solution branches of such hybrid equations are determined by the roots of a quadratic equation with coefficients depending upon the coefficients h_1 and h_2 of

Table 1. The six irreducible Painlevé transcendents.

Number	Painlevé transcendent	Dominant truncation	Balancing exponent	Resonances
$P_{\rm I}$	$u_{2x} = 6u^2 + x$	$u_{2x} = 6u^2$	-2	$-1, +6$
$P_{\rm II}$	$u_{2x} = 2u^3 + ux + a$	$u_{2x} = 2u^3$	-1	$-1, +4$
$P_{\rm III}$	$u_{2x} = u_x^2/u - u_x/u + (au^2+b)/x + cu^3 + d/u$	$u_{2x} = u_x^2/u + cu^3$	-1	$-1, +2$
$P_{\rm IV}$	$u_{2x} = u_x^2/2u + 3u^3/2 + 4u^2x + 2x^2 - 2a + b/u$	$u_{2x} = u_x^2/2u + 3u^3/2$	-1	$-1, +3$
$P_{\rm V}$	$u_{2x} = \left(\frac{1}{2u} + \frac{1}{u-1}\right)u_x^2 - u_x/x + \frac{(u-1)^2}{x^2}(au + b/u) + cu/x + eu(u+1)/u - 1$			
$P_{\rm VI}$	$u_{2x} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)u_x^2 - \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)u_x + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \times \left(a + \frac{bx}{u^2} + \frac{c(x-1)}{(u-1)^2} + \frac{ex(x-1)}{(u-x)^2}\right)$			

equation (11). Unless the two roots of this quadratic equation coincide, the corresponding hybrid equation (11) has two distinct solution branches. Thus the set of coefficients $h_1 = -1$ and $h_2 = +1$ for equation (11) leads to $a_0 = +1$ and $+2$ for the two solution branches when u is expressed as the power series in equation (4) and corresponds to the dominant truncation of the canonical Painlevé type equation X(table 2) which has 'semitranscendental' solutions [11]. Similarly the set of coefficients $h_1 = -3$ and $h_2 = -1$ for equation (11) leads to $a_0 = +1$ and -2 for the two solution branches and corresponds to the dominant truncation of the Painlevé canonical equation VI (table 2) which has solutions [11] of the form

$$u = -w_x / w \tag{12}$$

when $w(x)$ is the general solution of the linear equation of the third order.

$$w''' = q(x) w''. \tag{13}$$

Equation ($10d$) is passive; since it does not determine the balancing exponent p it can hybridise with any of the other building blocks, namely equations ($10a$)-($10c$).

Table 2. The first ten canonical Painlevé-type equations.

Number	Equation	Dominant truncation	Balancing exponent	Solution Resonances	type[a]
I	$u_{2x} = 0$				trivial
II	$u_{2x} = 6u^2$				elliptic
III	$u_{2x} = 6u^2 + \frac{1}{2}$	$u_{2x} = 6u^2$	-2	$-1, +6$	elliptic
IV	$u_{2x} = 6u^2 + x$				P_{I}
V	$u_{2x} = -2uu_x + q(x)u_x + r(x)u$	$u_{2x} = -2uu_x$	-1	$-1, +2$	semitrans
VI	$u_{2x} = -3uu_x - u^3 + q(x)(u_x - u^2)$	$u_{2x} = -3uu_x - u^3$	-1	$-1, +1$	
VII	$u_{2x} = 2u^3$				
VIII	$u_{2x} = 2u^3 + bu + c$	$u_{2x} = 2u^3$	-1	$-1, +4$	elliptic
IX	$u_{2x} = 2u^3 + ux + c$				P_{II}
X	$u_{2x} = -uu_x + u^3 - 12qu + 12r$	$u_{2x} = -uu_x + u^3$	-1	$-1, +3$[b] $-1, +6$[b]	semitrans

[a] Elliptic = solution expressed as elliptic functions, P_{I} and P_{II} = irreducible Painlevé transcendents (table 1), semitrans = semitranscendental.

[b] Two branches, each of which has -1 as the only non-positive resonance.

The dominant truncations of the last 40 of the 50 canonical Painlevé-type equations (i.e., XI to L in Ince's notation [11]) contain passive terms of the type u_x^2/u (i.e., equation $10d$) and will not be considered further here. The dominant truncations of the first ten canonical Painlevé-type equations contain only one or more of the active terms (equations ($10a$)-($10c$)) and are listed in table 2. In addition to the required -1, the resonances found in these equations include the positive integers 1, 2, 3, 4 and 6 for $p = -1$ and 6 for $p = -2$. Thus a variety of power series expansion behaviour is possible in even these relatively simple systems.

4. Painlevé chains

The following procedure can be used to generalise the concept of Painlevé chains introduced in the previous paper [10]. Consider a differential equation of order n

written in the following form:

$$f(x, u, u_x, u_{2x}, \ldots, u_{nx}) = g(x, u, u_x, u_{2x}, \ldots, u_{mx}) \qquad \text{where } m \leqslant n-1. \tag{14}$$

Furthermore, require equation (14) to have the following properties.

(1) Equation (14) has the Painlevé property with resonances $-1, r_2, \ldots, r_n$.

(2) Functions f and g are algebraic functions involving only sums, differences, products and quotients of their variables; transcendental functions such as exponentials, logarithms and trigonometric functions are not present in f and g.

(3) The function $f(x, u, u_x, u_{2x}, \ldots, u_{nx})$ includes all of the terms containing the highest derivative u_{nx} and no terms not containing u_{nx}; the equation $f = 0$ is thus an ordinary differential equation of order n.

(4) The function $g(x, u, u_x, u_{2x}, \ldots, u_{mx})$ contains no terms with the highest derivative u_{nx}; the equation $g = 0$ is thus an ordinary differential equation of order $m \leqslant n-1$ called the co-order [10] of equation (14).

(5) The function $g(x, u, u_x, u_{2x}, \ldots, u_{mx})$ cannot be obtained by differentiation of any function $h(x, u, u_x, u_{2x}, \ldots, u_{(m-1)x})$ which is an algebraic function of its variables.

Equation (14) defined in this manner can be the generator of a Painlevé chain where the members of the chain are obtained by successive and simultaneous differentiation of both sides of equation (14). Each differentiation step in a Painlevé chain retains all of the resonances of the previous members but adds one new resonance reflecting the position in the power series (equation (6)) of the new integration constant. In the relatively simple Painlevé chains discussed in this paper, where $f = u_{nx}$ in equation (14), the new resonance appears at $n-p-1$. When the new resonance duplicates an existing resonance, the Painlevé property is destroyed and the Painlevé chain is terminated. The number of equations in the Painlevé chain without double resonances is called its length and the order of the last member of the chain without a double resonance is called the order of the chain.

The previous paper [10] presents the three fundamental homogeneous Painlevé chains listed in table 3. The dominant truncations of the first two irreducible Painlevé transcendents, namely $u_{2x} = k_a u^2$ from P_I (equation (10a)) and $u_{2x} = k_c u^3$ from P_II

Table 3. The three fundamental homogeneous Painlevé chains.

2/2 Chain (p = −1; modified KdV; length 2; order 3)			
$u_{2x} = ku^3 \longrightarrow$	$u_{3x} = ku^2 u_x \dashrightarrow$	$u_{4x} = k(2uu_x^2 + u^2 u_{2x})$	
$(-1, +4)$	$(-1, +3, +4)$	$(-1, +3, +4, +4)$	
Dominant truncation of Painlevé P_II	Modified KdV	Double resonance (+4)	
1/1 Chain (p = −1; Burgers; infinite length and order)			
$u_x = ku^2 \longrightarrow$	$u_{2x} = kuu_x \longrightarrow$	$u_{3x} \times k(uu_{2x} + u_x^2) \longrightarrow$	$u_{4x} = k(uu_{3x} + 3u_x u_{2x}) \longrightarrow$
(-1)	$(-1, +2)$	$(-1, +2, +3)$	$(-1, +2, +3, +4)$
	Dominant truncation of Burgers		
2/1 Chain (p = −2, KdV; length 3; order 4)			
$u_{2x} = ku^2 \longrightarrow$	$u_{3x} = kuu_x \longrightarrow$	$u_{4x} = k(u_x^2 + uu_{2x}) \dashrightarrow$	$u_{5x} = k(3u_x u_{2x} + uu_{3x})$
$(-1, +6)$	$(-1, +4, +6)$	$(-1, +4, +5, +6)$	$(-1, +4, +5, +6, +6)$
Dominant truncation of Painlevé P_I	KdV	Boussinesq	Double resonance (+6)

(equation (10*c*)) generate the chains containing the modified KdV equation ($p=-2/2$) and KdV equation ($p=-2/1$) having orders 3 and 4, respectively. The other homogeneous and active dominant truncation of the second-order canonical Painlevé-type equations, namely $u_{2x}=k_b uu_x$ (equation (10*b*)), which is also the dominant truncation of Burger's equation, is the second member of a Painlevé chain ($p=-1/1$) of infinite order generated from the first-order differential equation $u_x=ku^2$.

The hybrid dominant truncations of the canonical Painlevé-type equations in table 2 can also be generators of Painlevé chains (table 4). Since these chains are hybrids, they have two solution branches although these branches can coalesce in exceptional cases (see the previous section). In the case of the solution branches of the Painlevé chain generated from $u_{2x}=-uu_x+u^3$ one solution branch, namely that with $a_0=1$ in equation (4), has only one member before +3 becomes a double root and thus has a length of only one whereas the other solution branch, namely that with $a_0=-2$ in equation (4), goes to order six before a double root (+6) arises. The hybrid Painlevé chain generated from $u_{2x}=-3uu_x-u^3$ has a solution branch, namely that with $a_0=+2$ in equation (4), which has a -2 resonance indicating that one of the integration constants cannot be accommodated in the power series expansion for u (equation (6)). The equations in the hybrid Painlevé chains in table 4, unlike those in the homogeneous Painlevé chains in table 3, do not appear to relate to differential equations of obvious physical significance.

Table 4. Hybrid Painlevé chains ($p=-1$).

Equation	Resonances	
	$a=1$	$a=\pm 2$
$u_{2x}=-uu_x+u^3$	$-1, +3$	$-1, +6$
$\downarrow$		
$u_{3x}=-u_x^2-uu_{2x}+3u^2u_x$	$-1, +3, +3$	$-1, +3, +6$
$\downarrow$		
$u_{4x}=-3u_xu_{2x}-uu_{3x}+6uu_x^2+3u^2u_{2x}$	$-1, +3, +3, +4$	$-1, +3, +4, +6$
$\downarrow$		
$u_{5x}=-uu_{4x}-4u_xu_{3x}-3u_{2x}^2+6u_x^3+18uu_xu_{2x}+3u^2u_{3x}$	$-1, +3, +3, +4, +5$	$-1, +3, +4, +5, +6$
$u_{2x}=-3uu_x-u^3$	$-1, +1$	$-1, -2$
$\downarrow$		
$u_{3x}=-3u_x^2-3uu_{2x}-3u^2u_x$	$-1, +1, +3$	$-1, -2, +3$
$\downarrow$		
$u_{4x}=-9u_xu_{2x}-3uu_{3x}-6uu_x^2-3u^2u_{2x}$	$-1, +1, +3, +4$	$-1, -2, +3, +4$
$\downarrow$		
$u_{5x}=-3uu_{4x}-12u_xu_{3x}-9u_{2x}^2-6u_x^3-18uu_xu_{2x}-3u^2u_{3x}$	$-1, +1, +3, +4, +5$	$-1, +2, +3, +4, +5$

Passive Painlevé chains can also be generated from the Painlevé canonical equations. Consider the following pure passive second-order differential equation:

$$u_{2x}=ku_x^2/u. \tag{15}$$

The value of k determines the value of p in the expansion in equations (4) and (6) according to the relationship

$$k=(p-1)/p. \tag{16}$$

Table 5. The passive Painlevé chains.

Equation	Balancing exponent, p	Resonances
$uu_{2x}=2u_x^2$	-1	$-1, 0$
$\downarrow$		
$uu_{3x}=3u_xu_{2x}$	-1	$-1, 0, +4$
$\downarrow$		
$uu_{4x}=3u_{2x}^2+2u_xu_{3x}$	-1	$-1, 0, +4, +5$
$\downarrow$		
$uu_{5x}=u_xu_{4x}+8u_{2x}u_{3x}$	-1	$-1, 0, +4, +5, +6$
$\downarrow$		
$uu_{6x}=9u_{2x}u_{4x}+8u_{3x}^2$	-1	$-1, 0, +4, +5, +6, +7$
$2uu_{2x}=3u_x^2$	-2	$-1, 0$
$\downarrow$		
$uu_{3x}=2u_xu_{2x}$	-2	$-1, 0, +6$
$\downarrow$		
$uu_{4x}=u_xu_{3x}+2u_{2x}^2$	-2	$-1, 0, +6, +7$
$\downarrow$		
$uu_{5x}=5u_{2x}u_{3x}$	-2	$-1, 0, +6, +7, +8$

Table 5 shows the Painlevé chains derived from equation (15) for $p=-1(k=2)$ and $p=-2(k=\frac{3}{2})$. Note the infinite lengths of these chains and the zero resonances arising from the passive determination of the balancing exponent p. The new resonances in the third-order equations of these Painlevé chains appear at $2-2p$ and the new resonances in the nth-order equations appear at $2-2p+n-3=n-2p-1$.

5. A Schwarzian Painlevé chain

The Schwarzian derivative $\{u;\, x\}$ is defined by the following expression:

$$\{u;\, x\}=(u_{3x}/u_x)-\tfrac{3}{2}(u_{2x}/u_x)^2. \tag{17}$$

It is significant in being invariant under the Möbius transformation

$$u'=(au+b)/(cu+d). \tag{18}$$

Weiss [12-15] has related the Schwarzian derivative to integrable partial differential equations including the Burger's, KdV, modified KdV and Boussinesq equations.

Table 6 shows how a Painlevé chain can be obtained from the Schwarzian derivative (equation (17)). Set the Schwarzian derivative $\{u;\, x\}$ equal to zero and for convenience

Table 6. The Schwarzian Painlevé chain ($p=-1$).

Equation	Resonances
$2u_xu_{3x}=3u_{2x}^2$	$-1, 0, +1$
$\downarrow$	
$u_x^2u_{4x}=4uu_{2x}u_{3x}-3u_{2x}^3$	$-1, 0, +1, +2$
$\downarrow$	
$u_x^3u_{5x}=5u_{4x}u_{2x}u_x^2+4u_{3x}^2u_x^2-17u_{3x}u_{2x}^2u_x-9u_{2x}^4$	$-1, 0, +1, +2, +3$

multiply both sides by u_x^2 to clear the terms in the denominators of equation (17) thereby giving the first member of the Painlevé chain. The balancing exponent p is determined passively to be -1 so that the $\frac{3}{2}$ coefficient is required for its determination. This third-order differential equation has the consecutive resonances $r=-1$, 0, +1, with the zero resonance arising from the passive nature of this equation. In order to obtain successive members of the Schwarzian Painlevé chain (table 6), divide both sides by u_x^{n-1}, differentiate both sides and multiply both sides by u_x^n to clear the terms in the denominator where n is the order of the starting differential equation. The resulting Painlevé chain is of infinite length and a member of the chain of order n has consecutive resonances $r=-1, 0, +1, \ldots, n-2$, indicating that all of the integration constants might appear in the first $n-1$ consecutive terms of a Laurent series (equation (6)) if compatibility conditions are satisfied at the resonances. This Schwarzian Painlevé chain contains the dominant truncations of the integrable class of partial differential equations

$$u_t/u_x + B(\{u; x\}) = 0 \tag{19}$$

in which B is a constant coefficient multinomial in $(\partial^k/\partial x^k)\{u; x\}$. Weiss [14] has shown that these higher-order differential equations arising from the Schwarzian derivative are useful for generating the higher-order KdV and other evolution equations of interest.

6. Chains from eigenvalue problems

The Painlevé chains containing the KdV and modified KdV equations each have finite lengths owing to the appearance of double roots in the indicial equation (8) used to determine the resonances. These double roots can be eliminated by hybridisation with an appropriately chosen equation having the same balancing p or with a pure passive equation. For example, the fifth-order differential equation in the KdV Painlevé chain ($p=-2/1$) [10] namely

$$u_{5x} = k(3u_x u_{2x} + uu_{3x}) \tag{20}$$

has the resonances $-1, +4, +5, +6, +6$ (table 3). The double root (+6) can be eliminated by hybridisation [10] with the following equation:

$$u_{5x} = 3mu^2 u_x. \tag{21}$$

Equation (21) arises from the ($p=-4/2$) chain generated by differentiating

$$u_{4x} = mu^3. \tag{22}$$

In the homogeneous form equation (22) does not have the Painlevé property since its indicial equation (8) has complex roots corresponding to complex resonances.

A hybrid fifth-order higher KdV equation is well known [20] to have the following form:

$$u_{5x} = -20u_x u_{2x} - 10\, uu_{3x} - 30u^2 u_x. \tag{23}$$

Both solution branches of this hybrid equation have distinct integral resonances. The branch leading to the resonances $-1, +2, +5, +6, +8$ is the most significant since all of the (integral) resonances are greater than -1 indicating the possibility of incorporating the integration constants into a Laurent expansion of u (equation (6)).

How can one determine the coefficients of hybrid equations such as equation (23) which can have the Painlevé property? Consider the second-order eigenvalue problem [3]

$$v_x = -\mathrm{i}\lambda v + qw \tag{24a}$$

$$w_x = rv + \mathrm{i}\lambda w \tag{24b}$$

$$v_t = Av + Bw \tag{24c}$$

$$w_t = Cv - Aw. \tag{24d}$$

Compatibility of conditions (24*a*)-(24*d*) requires the following equations to be satisfied:

$$A_x = qC - rB \tag{25a}$$

$$B_x + 2\mathrm{i}\lambda B = q_t - 2Aq \tag{25b}$$

$$C_x - 2\mathrm{i}\lambda C = r_t + 2Ar. \tag{25c}$$

In order to obtain a chain of higher-order differential equations of potential interest, substitute the following power series for A, B and C into equations (25*a*)-(25*c*) taking the integer n as high as the order of that of the desired differential equation:

$$A = \sum_{k=0}^{n} A_k \lambda^k \tag{26a}$$

$$B = \sum_{k=0}^{n} B_k \lambda^k \tag{26b}$$

$$C = \sum_{k=0}^{n} C_k \lambda^k. \tag{26c}$$

Equate the coefficients of like powers of λ in equations (25*a*)-(25*c*) and solve the resulting $3n+5$ equations. This procedure involves determining the $3n+3$ coefficients of the power series (26*a*)-(26*c*) in the following sequence:

$$B_n, C_n, A_n, B_{n-1}, C_{n-1}A_{n-1}, \ldots, B_0, C_0, A_0. \tag{27}$$

Determination of the coefficients B_k and C_k $(n-2 \geqslant k \geqslant 0)$ requires differentiations of B_{k+1} and C_{k+1}, respectively, whereas determining each $A_k (n \geqslant k \geqslant 0)$ requires an integration. The latter lead to the n integration constants designated as a_k $(n \geqslant k \geqslant 0)$. After all $3n+3$ coefficients (27) are determined from the first $3n+3$ equations, substituting these coefficients into the last two of the original $3n+5$ equations (derived from equations (25*b*) and (25*c*)) gives equations in q_t or r_t, respectively, in which all of the other terms contain exactly one of the integration constants a_k. Grouping together the terms containing a given a_k while setting $a_j = 0$ for all $j \neq k$ and adding a new equation either relating q to r or setting q or r to a constant gives a kth-order differential equation which in certain cases corresponds to known evolution equations [3]. For example $k=3$, $r=-1$ and $a_j = 0$ for $j \neq 3$ give the KdV equation in q whereas $k=3$, $r=q$ and $a_j=0$ for $j \neq 3$ give the modified KdV equation. This procedure is described in greater detail elsewhere [3].

Table 7 shows what happens when this procedure is done with fifth degree polynomial expansions of A, B and C ($n=5$ in equations (26*a*)-(26*c*)) substituted in equations (25*a*)-(25*c*) using the symmetrical relationship $q=r$ after considering the time-independent situation where $q_t = 0$. The symmetry of the relationship $q=r$ makes the

Table 7. A time-independent chain from the 2×2 eigenvalue problem ($r=q, p=-1$).

Equation	Resonances[a]	Comments[b]
$q_{2x}=2q^3$	$-1,+4$	DT of P_{II}
$q_{3x}=6q^2q_x$	$-1,+3,+4$	DT of mKdV
$q_{4x}=10q^2q_{2x}+10q_x^2q-6q^5$	$-1,+2,+3,+6$	
$q_{5x}=10q^2q_{3x}+40qq_xq_{2x}+10q_x^3-30q^4q_x$	$-1,+2,+3,+5,+6$	DT of fifth-order mKdV

[a] When there are two solution branches (i.e. for the fourth- and fifth-order equations which are hybrids) the solution branch having no resonances below -1 is chosen.
[b] DT = dominant truncation.

final two equations identical for each member of the chain. The equations in table 7 are identical with the modified KdV chain in table 3 for $k\leqslant 3$. For $k\geqslant 4$ hybridisation is automatically introduced by this procedure. These hybrids, at least for $k=4$ and $k=5$, have integral resonances. The fifth-order modified KdV equation was discovered by Ito [21] using a different method; this equation can also be obtained by differentiation of the fourth-order equation above it in the modified KdV chain (table 7).

Generation of the KdV equation by an analogous method requires the relationship $r=-1$ which treats the variables q and r asymmetrically. This reduces the final two of the $3n+5$ equations from (26*a*)-(26*c*) and (25*a*)-(25*c*) to a single equation only if k is odd. The resulting single equations obtained for $k=3$ and $k=5$ are the dominant truncations of the KdV equation and the fifth-order KdV equation [20], respectively. The absence of even-order differential equations in the KdV chain generated by this method is consistent with results obtained by using recursion [22] or differential [23] operators for finding higher-order KdV equations.

These observations suggest that the chain of KdV equations contains only odd-order members whereas the chain of modified KdV equations contains both even- and odd-order members.

7. Summary

The previous paper [10] defines the three fundamental homogeneous active Painlevé chains (table 3) which are generated from the equations $u_{2x}=ku^2$, $u_{2x}=ku^3$ and $u_x=ku^2$. The present paper shows how the concept of Painlevé chains can be extended to Painlevé chains generated from hybrid differential equations (table 4) and the passive differential equation $u_{2x}=ku_x^2/u$ (table 5). In addition the Schwarzian derivative can be used to generate a Painlevé chain with the interesting property of consecutive integral resonances (table 6). Finally, the 2×2 eigenvalue problem of Zakharov and Shabat [16] is shown to generate chains exhibiting some but not all of the features of Painlevé chains.

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